This article explores the forced vibrations of inhomogeneous nano-micro elements based on a non-local theory of elasticity. Here, as a structural element, we take a straight rod and the Euler – Bernoulli rod theory is adopted for it. It is assumed that the elastic modulus of the material of the rod is a continuous function of the thickness coordinate. When solving the problem, the equations of oscillations obtained on the basis of the Euler – Bernoulli rod theory using the equation of state of the non-local theory of elasticity proposed by Eringen were used. Analytical formulas for determining the deflection of a rod are obtained for various cases of boundary conditions.

**Keywords:** inhomogeneous nano-microelement, theory of Euler – Bernoulli rods, stability, critical load, non-local theory of elasticity.

**Introduction.** In modern mechanics, the creation of materials with a nanoscale structure (nanomaterials) and composite materials filled with nano-objects occupies an important place. Such materials have unique properties (sorption, electrical, transport and mechanical). Unusual mechanical properties of nanomaterials have a strong influence on their other physical characteristics, and therefore attract much attention. This applies, above all, to such nano-objects as carbon and non-carbon nanotubes and nanos, graphenes, thin films, nanoslands (quantum dots), nanoclusters, nanocomposites. There are a large number of papers in which carbon nanotubes have been shown to possess a number of features and have various applications. The theory of nonlocal continuum mechanics was introduced by A. Eringen [1]. According to this theory, stress at a point is a function of stress at all points of the body. In [9], an elastic longitudinal bending of a long two-layer carbon nanotube placed in an elastic medium and under the action of a far-field hydrostatic pressure was studied using the energy method. It is shown that the critical pressure depends on the ratio of the internal radius to thickness, the parameters of the material of the elastic medium, and the van der Waals force. In [2], using models of Euler – Bernoulli and Timoshenko beams and models of Donnel shells, transverse vibrations of single- and two-layer carbon nanotubes under axial load were investigated. It has been established that the predictions made using the Euler-Bernoulli beam model and the Donnel shell model are the least and most accurate, respectively. An approximate method for studying dynamic buckling of two-layer carbon nanotubes under sudden axial loading was proposed in [8]; a model of elastic shells was developed for studying dynamic buckling of two-layer carbon
nanotubes under step axial loading. This model takes into account the van der Waals forces between the outer and inner tubes. The results showed that with dynamic buckling, the critical load for double-layer carbon nanotubes is greater than in the case of static buckling. In [10], using a non-local theory of elasticity, a non-local model of multilayer shells was proposed in the critical load problem for a multi-layered carbon nanotube with uniform external radial pressure. An explicit expression for the critical pressure was obtained, and the effect of a small length scale on the critical pressure was investigated. In [3, 4, 6], as a structural element, take a straight rod and adopted the Euler–Bernoulli rod theory for it. It is assumed that the modulus of elasticity of the material of the rod is a continuous function of the thickness coordinate. In obtaining the stability equations based on the Euler-Bernoulli rod theory, the equations of state of the nonlocal theory of elasticity proposed by Eringen were used. For various cases of boundary conditions, the stability equations for the considered rods are obtained. After solving the obtained equations, analytical formulas were found to determine the critical load and various analyzes were performed. The stability of inhomogeneous rods based on the non-local Eringen theory is studied in [7, 5].

**Setting and solving the problem.** To study the forced vibrations of inhomogeneous nano-micro elements according to the non-local theory of Eringen, we will use the following equations:

\[
\frac{\partial^2}{\partial x^2} \left( -KI \frac{\partial^2 w}{\partial x^2} \right) + \mu \frac{\partial^2}{\partial x^2} \left[ \frac{\partial}{\partial x} \left( P \frac{\partial w}{\partial x} \right) - q + m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] + \\
+ q - \frac{\partial}{\partial x} \left( P \frac{\partial w}{\partial x} \right) = m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2}.
\]

(1)

Here, \( m_0 = \int \rho ds = \rho s \); \( m_2 = \int z^2 ds = \rho s \frac{h^2}{12} \), \( q \) — force acting on the surface of the bar. We assume that the material is non-uniform, that is, the elastic modulus is a continuous function of the thickness coordinate: \( E = E(z) \). \( KI \) generalized characteristic stiffness.

If \( E \) changes in the form \( E = E_0 \left( 1 + \gamma \frac{z^4}{h^4} \right) \), in this case \( E = E(z) \),

\( KI = E_0 I \left( 1 + \gamma \frac{3}{112} \right) \). He \( E_0 I \) is the stiffness of the corresponding homogeneous bar. We will assume that only a constant axial force of compression acts on a bar, i.e. in equation (1) is \( P = \text{const} \). Suppose that a constant axial compression force acts on the bar, and that the forced forces on the surface
are harmonically variable in time \( q = q_0 e^{i\omega t} \). Considering the above, equation (1) can be written as follows:

\[
\left( \mu P - KI \right) \frac{\partial^4 w}{\partial x^4} - P \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2}{\partial x^2} \left[ m_0 \frac{\partial^2 w}{\partial t^2} - m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} \right] - m_0 \frac{\partial^2 w}{\partial t^2} + m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} = \mu \frac{\partial^2 q}{\partial x^2} - q.
\]  

(2)

Since equation (2) is inhomogeneous, its general solution consists of solutions of the corresponding homogeneous equation and one particular solution of the inhomogeneous equation. The solution of the homogeneous equation corresponding to equation (2) will be sought in the following form:

\[
w(x,t) = w_1(x)e^{i\omega t}.
\]  

(3)

Writing expression (3) in the homogeneous equation corresponding to equation (2) we get

\[
d_1 \frac{d^4 w_1}{dx^4} + d_2 \frac{d^2 w_1}{dx^2} - rw_1 = 0.
\]  

(4)

Here are the following replacements:

\[
d_1 = KI - \mu P - \omega m_2 \omega^2, \quad d_2 = m_2 \omega^2 + \mu m_0 \omega^2, \quad r = m_0 \omega^2,
\]  

(5)

\( \omega \) – oscillation frequency.

The general solution of equation (4) is:

\[
w_1 = c_1 \sin \alpha x + c_2 \cos \alpha x + c_3 \sinh \beta x + c_4 \cosh \beta x.
\]  

(6)

Here \( c_1, c_2, c_3, c_4 \) – integral constants determined from boundary conditions;

\[
\alpha^2 = \frac{1}{2d_1} \left( d_2 + \sqrt{d_2^2 + 4d_1 r} \right); \quad \beta^2 = \frac{1}{2d_1} \left( -d_2 + \sqrt{d_2^2 + 4d_1 r} \right).
\]  

(7)

In the particular case, if \( a = b, P = 0, m_2 = 0, \mu = 0 \), you can get a solution to this problem according to the classical local theory of elasticity.

We look at one particular solution of equation (2) as follows:

\[
w_{10} = \tilde{w} e^{i\omega t}.
\]  

(8)
The expression (8) is entered in equation (2) and taking into account that \( q = q_se^{i\omega t} \) for \( \tilde{w} \) we get

\[ \tilde{w} = -\frac{q_s}{m_0}. \]

As a result, for the general solution of equation (2), we obtain:

\[ w_1 = c_1 \sin \alpha x + c_2 \cos \alpha x + c_3 \sinh \beta x + c_4 \cosh \beta x - \frac{q_s}{m_0}. \]  

(9)

Making some transformations using the expression (9) we get

\[ \frac{dw_1}{dx} = \alpha \left( c_1 \cos \alpha x - c_2 \sin \alpha x \right) + \beta \left( c_3 \sinh \beta x + c_4 \cosh \beta x \right); \]  

(10)

\[ M = -d_1 \frac{d^2w}{dx^2} - \mu w_1 = \left( d_1 \alpha^2 - \mu r \right) \left( c_1 \sin \alpha x + c_2 \cos \alpha x \right) - \left( d_1 \beta^2 + \mu r \right) \left( c_3 \sinh \beta x + c_4 \cosh \beta x \right) + \mu r \frac{q_s}{m_0}; \]  

(11)

\[ N = -d_1 \frac{d^3w_1}{dx^3} - d_2 \frac{dw_1}{dx} = \alpha \left( d_1 \alpha^2 - d_2 \right) \left( c_1 \cos \alpha x - c_2 \sin \alpha x \right) - \beta \left( d_1 \beta^2 + d_2 \right) \left( c_3 \sinh \beta x + c_4 \cosh \beta x \right). \]  

(12)

To determine the constant integrals \( c_1, c_2, c_3, c_4 \) we consider two different boundary conditions.

First consider the case of hinge mounting both ends of the bar. In this case, when \( x = 0 \) and \( x = a \), the following boundary conditions should be satisfied:

\[ w_1 = 0 \quad \text{and} \quad M = -d_1 \frac{d^2w_1}{dx^2} - \mu m_0 \omega^2 w_1 = 0, \]  

(13)

or the condition must be satisfied

\[ w_1 = 0 \quad \text{and} \quad \frac{d^2w_1}{dx^2} = 0, \]  

(14)

equivalent to conditions (13).

If we use (9), from conditions (14) we get:
\[ c_2 + c_4 = \frac{q_s}{m_0}; \quad c_1 \sin \alpha a + c_2 \cos \alpha a + c_3 \sinh \beta a + c_4 \cosh \beta a = \frac{q_s}{m_0}; \]

\[ \alpha^2 c_2 - \beta^2 c_4 = 0; \quad -\alpha^2 \sin \alpha c_1 - \alpha^2 \cos \alpha c_2 + \beta^2 \sinh \beta c_3 + \beta^2 \cosh \beta c_4 = 0. \]

The system of equations (15) is a system of inhomogeneous linear equations with respect to the integral constants \( c_1, c_2, c_3, c_4 \). This system allows you to define these constants uniquely:

\[ c_1 = \frac{\Delta_1}{\Delta}; \quad c_2 = \frac{\Delta_2}{\Delta}; \quad c_3 = \frac{\Delta_3}{\Delta}; \quad c_4 = \frac{\Delta_4}{\Delta}. \]

Here \( \Delta \) is the main determinant, and \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) are the auxiliary determinants of the system:

\[ \Delta = \left( \alpha^2 + \beta^2 \right)^2 \sin \alpha \sinh \beta a; \quad \Delta_1 = \frac{q_s}{m_0} \left( \alpha^2 + \beta^2 \right) \beta \left( \cos \alpha a - 1 \right) \sin \alpha \sinh \beta a; \]

\[ \Delta_2 = - \frac{q_s}{m_0} \left( \alpha^2 + \beta^2 \right) \sin \alpha \sinh \beta a; \quad \Delta_3 = \frac{q_s}{m_0} \left( \alpha^2 + \beta^2 \right) \alpha^2 \sin \alpha \left( \cosh \beta a + 1 \right); \]

\[ \Delta_4 = - \frac{q_s}{m_0} \left( \alpha^2 + \beta^2 \right) \alpha^2 \sin \alpha \sinh \beta a. \]

As a result, for integral constants \( c_1, c_2, c_3, c_4 \), we obtain:

\[ c_1 = - \frac{q_s}{m_0} \frac{\beta \left( \cos \alpha a - 1 \right)}{\left( \alpha^2 + \beta^2 \right) \sin \alpha a}; \quad c_2 = - \frac{q_s}{m_0} \frac{1}{\alpha^2 + \beta^2}; \]

\[ c_3 = - \frac{q_s}{m_0} \frac{\cosh \beta a + 1}{\left( \alpha^2 + \beta^2 \right) \sinh \beta a}; \quad c_4 = \frac{q_s}{m_0} \frac{\alpha^2}{\alpha^2 + \beta^2}. \]

If in equation (9) we write these expressions constant, for the deflection of the bar we get

\[ w_1 = - \frac{q_s}{m_0} \left[ \frac{1}{\alpha^2 + \beta^2} \left( \frac{\beta \left( \cos \alpha a - 1 \right)}{\sin \alpha a} \sin \alpha x + \cos \alpha x + \frac{\cosh \beta a + 1}{\sinh \beta a} \sinh \beta x - \alpha^2 \cosh \beta x \right) + 1 \right] \quad (16) \]

Now consider the case of rigid mounting both ends of the bar. In this case, when \( x = 0 \) and \( x = a \), the boundary conditions will be as follows:

\[ w = 0 \quad \text{and} \quad \frac{dw}{dx} = 0. \quad (17) \]
If we use expressions (9) and (10) we get:

\[
c_2 + c_4 = \frac{q_s}{m_0}; \quad c_1 \sin \alpha a + c_2 \cos \alpha a + c_3 \sinh \beta a + c_4 \cosh \beta a = \frac{q_s}{m_0}; \quad (18)
\]

\[\alpha (c_1 \cos \alpha a - c_2 \sin \alpha a) + \beta (c_3 \cosh \beta a + c_4 \sinh \beta a) = 0; \quad \alpha c_1 + \beta c_3 = 0;
\]

From system (18) we define the integral constants \(c_1, c_2, c_3, c_4:\)

\[c_1 = \frac{\Delta_1}{\Delta}; \quad c_2 = \frac{\Delta_2}{\Delta}; \quad c_3 = \frac{\Delta_3}{\Delta}; \quad c_4 = \frac{\Delta_4}{\Delta}.
\]

Here \(\Delta\) is the main determinant, and \(\Delta_1, \Delta_2, \Delta_3, \Delta_4\) auxiliary determinants of the system (18):

\[
\Delta = \left(\beta^2 - \alpha^2\right) \sin \alpha \sinh \beta a + 2 \alpha \beta;
\]

\[
\Delta_1 = -\frac{q_s}{m_0} \beta (\cos \alpha a - 1) \sinh \beta a + \alpha \sin \alpha a (\cosh \beta a - 1);
\]

\[
\Delta_2 = -\frac{q_s}{m_0} (\alpha \beta (\cos \alpha a + 1) (\cosh \beta a - 1) - \beta^2 \sin \alpha \sinh \beta a);
\]

\[
\Delta_3 = \frac{q_s}{m_0} (\alpha \beta \sinh \beta a (\cos \alpha a + 1) + \alpha^2 \sin \alpha a (\cosh \beta a - 1));
\]

\[
\Delta_4 = \frac{q_s}{m_0} (\alpha \beta \cos \alpha a (\cosh \beta a - 1) + \alpha^2 \sin \alpha \sinh \beta a).
\]

As a result, for integral constants \(c_1, c_2, c_3, c_4\) we obtain:

\[
c_1 = \frac{-\frac{q_s}{m_0} \beta (\cos \alpha a - 1) \sinh \beta a + \alpha \sin \alpha a (\cosh \beta a - 1)}{\left(\beta^2 - \alpha^2\right) \sin \alpha \sinh \beta a + 2 \alpha \beta};
\]

\[
c_2 = \frac{-\frac{q_s}{m_0} (\alpha \beta (\cos \alpha a + 1) (\cosh \beta a - 1) - \beta^2 \sin \alpha \sinh \beta a)}{\left(\beta^2 - \alpha^2\right) \sin \alpha \sinh \beta a + 2 \alpha \beta};
\]

\[
c_3 = \frac{\frac{q_s}{m_0} (\alpha \beta \sinh \beta a (\cos \alpha a + 1) + \alpha^2 \sin \alpha a (\cosh \beta a - 1))}{\left(\beta^2 - \alpha^2\right) \sin \alpha \sinh \beta a + 2 \alpha \beta};
\]

\[
c_4 = \frac{\frac{q_s}{m_0} (\alpha \beta \cos \alpha a (\cosh \beta a - 1) + \alpha^2 \sin \alpha \sinh \beta a)}{\left(\beta^2 - \alpha^2\right) \sin \alpha \sinh \beta a + 2 \alpha \beta}.
\]
\[
\begin{align*}
c_4 &= \frac{q_s}{m_0} \left( \alpha \beta \cos \alpha a (\sin \beta a - 1) + \alpha^2 \sin \alpha sh \beta a \right) \\
&\quad \quad \quad \quad \quad \frac{\left( \beta^2 - \alpha^2 \right) \sin \alpha sh \beta a + 2\alpha \beta}{.}
\end{align*}
\]

If in equation (9) we write these expressions constant, for the deflection of the bar we get

\[
w_1 = -\frac{q_s}{m_0} \left[ \alpha (\sin \beta a - 1) (\beta \sin \alpha \sin \alpha x - \alpha \sin \alpha sh \beta x - \beta \cos \alpha ach \beta x + \\
\quad \quad \quad \quad \quad \beta^2 (\cos \alpha a - 1) \sin \alpha x - \left( \beta^2 \cos \alpha x + \alpha^2 \sin \alpha \sin \beta x \right) \sin \alpha a \sin \beta a \right]
\]

\[
\left( \beta^2 - \alpha^2 \right) \sin \alpha sh \beta a + 2\alpha \beta \right]^{-1}. \tag{19}
\]

**Conclusion.** Explores the forced vibrations of inhomogeneous nano-micro elements based on a non-local theory of elasticity. The equation of state of the non-local theory of elasticity proposed by Eringen were used. Analytical formulas for determining the deflection of a rod are obtained for various cases of boundary conditions.

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В. Г. Раджабов

ВИМУШЕНІ КОЛІВАННЯ НЕОДНОРІДНИХ НАНО-МІКРОЕЛЕМЕНТІВ ЗА НЕЛОКАЛЬНОЮ ТЕОРІЄЮ ЕРІНГЕНА

Досліджуються вимушені коливання неоднорідних нано-мікроелементів на основі нелокальної теорії пружності. Як елемент конструкції взято прямолінійний стержень і для нього прийнята теорія стрижнів Ейлера – Бернуллі. Передбачається, що модуль пружності матеріалу стержня є неперервною функцією координати товщиної. При розв’язуванні задачі використані рівняння коливань, отримані на основі теорії стрижнів Ейлера – Бернуллі з використанням рівняння стану нелокальної теорії пружності, що запропоновано Ерінгеном. Для різних випадків граничних умов отримано аналітичні формулі для визначення проги- ну стрижня.

Ключові слова: неоднорідний нано-мікроелемент, теорія стрижнів Ейлера – Бернуллі, стійкість, критичне навантаження, нелокальна теорія пружності.